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# The spin glass random matrix ensemble: some inequalities for a new matrix norm

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**Abstract.** In spin glasses with quenched position randomness the interaction matrix $\mathbf{J} = \{J_{ij}\}$ generates a random matrix ensemble, distinguished by correlations between the matrix elements, as compared to simplified models with independent $J_{ij}$ corresponding to more conventional random matrix ensembles. Hardly anything seems to be known about the former ensemble, which is the more realistic one. There are reasons to believe that the unconventional matrix norm

$$\|\mathbf{J}\| = \frac{1}{n}\sum_i \left( \sum_{j(\neq i)} J_{ij}^2 \right)^{1/2}$$

may be of special significance. Several inequalities are proved for this norm and its ensemble mean $\langle\|\mathbf{J}\|\rangle$. In particular, it is shown that $\|\mathbf{J}\| \geqslant |\det(\mathbf{J})|^{1/n}$ and $\langle\|\mathbf{J}\|\rangle \geqslant 1/2ew_{\mathrm{m}}$, where for a continuous ensemble eigenvalue density $w(x) = w(-x)$ of spectral range $|x| \leqslant \Delta$: $w_{\mathrm{m}} = \max(w(x), 1/\Delta)$. The new norm is also related by inequalities to the ordinary Euclidean norm. These results indicate that the present matrix ensemble has several remarkable features, notably a sharp peak in $w(x)$ at $x = 0$, which appear to rule out a conventional Wigner density $w(x)$. A discussion is given of related topics, such as quenched against ergodic randomness, matrix ergodicity (i.e. $n \to \infty$), spin glass $c$ scaling and space dilatation.

Real spin glass systems have a definite space configuration of spin-carrying atoms, in which a random distribution of positions is obtained by rapid quenching of a molten mixture of these atoms with non-magnetic atoms. When the system is a metallic alloy, which is the case with the majority of the many systems that have been experimentally investigated, the spins can interact with each other at long range by means of the RKKY indirect exchange mechanism. Let the spin operators belonging to two sites, $i$ and $j$, indicating the set of quenched random positions, be $\vec{\mathbb{S}}_i$ and $\vec{\mathbb{S}}_j$. Then the Hamiltonian is

$$\mathbb{H} = -\sum_{ij} J_{ij} \vec{\mathbb{S}}_i \cdot \vec{\mathbb{S}}_j. \tag{1}$$

In a space of three dimensions the interaction constants $\{J_{ij}\}$ are determined by the distances $\{R_{ij}\}$ between the spin sites and has the form

$$J_{ij} = J_0 \frac{\cos(R_{ij}/a_0)}{(R_{ij}/a_0)^3} \tag{2}$$

where $a_0$ is an atomic length scale and $J_0$ is an energy scale. This interaction has been investigated in considerable mathematical detail (Ruderman and Kittel 1954, Kasuya 1956, Yosida 1957, Kaneyoshi 1975, Fischer and Klein 1975, Larsen 1980, 1981, 1985, 1987) and is known to be the cause of the spin glass effects that are universally observed in these systems. The randomness in position translates into a random sign of the different exchange couplings $J_{ij}$. The distribution of magnitudes of the $J_{ij}$ depends on the fractional concentration $c = N_s/N_a$ of spin-carrying atoms, where $N_s$ is the number of spins and $N_a$ is the number of both kinds of atoms, because of the distance dependence in (2).

The statistical mechanics of the spin glass category remains an unsolved problem. Considerable theoretical effort has recently been directed towards simplified models, in which the spins are frequently replaced by classical binary variables (Ising spins) on a filled lattice ($c = 1$), and where the interactions $\{J_{ij}\}$ are random variables with some simplified *a priori* distribution. Although such models definitely capture much of the essence of the problem, it is not known to what extent the results that have been obtained apply to the original model (1) and (2). About the statistical mechanics of the latter, very little is known with any degree of mathematical certainty.

One of the earliest, and most well established, experimental results in the real metallic spin glass systems is the so-called $c$ scaling (Soulette and Tournier 1969, Liu and Smith 1975). Let $M$ be some observable quantity, such as the magnetisation. Originally the $c$ scaling between different spin glass systems, with the same atomic constitution but different concentrations $c$, was cast in the form

$$M(c, T, H) = c^z f(T/c, H/c) \tag{3}$$

where $T$ is the temperature, $H$ the magnetic field and $z = 0$ for an intensive quantity or $z = 1$ for an extensive one. Subsequently it has been observed that the scaling is significantly improved if, instead of $c$, one scales with a $c$-dependent energy scale, $g(c)$, i.e.

$$M(c, T, H) = c^z F(k_B T/g(c), \mu_B H/g(c)) \tag{4}$$

where $g(c)$ is taken from the data, say in the form of a characteristic freezing temperature, $k_B T_0$ (Larsen 1978). This allows for deviations from strict linearity ($g(c) \propto c$), the most significant cause being imperfections in the electronic medium which transmits the RKKY interaction (2). This is known to supply the ideal range dependence in (2) with an exponential damping factor, restricting the range to distances inside an electronic mean free path. While this does not affect the scaling of the spin glass effects, it modifies $g(c)$ in a significant way. Possibly this damping effect is the most direct evidence for the realistic character of the RKKY-based model†.

As the scaling suggests, presuming it reflects an essential aspect of the statistical mechanics of the spin glass model (1) and (2), the energy scale $g(c)$ must be a functional of the interaction matrix $\mathbf{J} = \{J_{ij} \mid J_{ii} = 0, \forall i\}$ (which functional may or may not depend on the spin magnitude). Even as considered apart from the unsolved statistical mechanics problem, the consideration of such functionals raises some interesting questions. We shall, for simplicity, assume that the random sign variation in $\mathbf{J}$ disqualifies linear functionals. It might be thought straightforward to pick a second-order one, such as the Hilbert-Schmidt norm (divided by $N_s^{1/2}$)

$$\|\mathbf{J}\| = \left(\frac{1}{N_s} \sum_{i,j(\neq i)} J_{ij}^2\right)^{1/2}. \tag{5}$$

† Because doubts have recently been expressed in the literature as to whether the RKKY interaction can become modified so as to have only a finite range, a discussion of this point has been requested. Although the question has no direct implications for the present work, it belongs to the context in which it is situated. A brief discussion in general terms is provided in appendix 3, but for details we must refer the interested reader to the literature.

But, as we have pointed out, because of the quenched randomness inherent in the matrix **J** at $c<1$, this is not so; the functionals depend in a significantly different way on $c$ (Larsen 1977, 1986b). An alternative choice is

$$\|\mathbf{J}\|_{\text{quenched}}=\frac{1}{N_{\mathrm{s}}}\sum_{i}\left(\sum_{j(\neq i)}J_{ij}^{2}\right)^{1/2}. \tag{6}$$

It is easy to see that this is also a matrix norm†. In the following we establish two chains of inequalities pertaining to these norms. The second of these concerns the averages on ensembles of random matrices, such as are generated, for instance, when different **J** are created on the basis of different quenched random spin configurations. These results are then to be applied to the specific $c$-scaling question arising in the context of the real spin glass systems and establish a rigorous background for some approximate and numerical estimates that have been proposed (Larsen 1977, 1987, Riess and Ron 1973).

*Theorem 1.* Let $\mathbf{A}=\{a_{ij}\}$ be a Hermitian $n\times n$ matrix over the complex number field, and let $\det(\mathbf{A})$ be its determinant. Let two norms, referred to as the 'ergodic' and the 'quenched', be defined by

$$\|\mathbf{A}\|_{\text{ergodic}}=\left(\frac{1}{n}\sum_{i,j}|a_{ij}|^{2}\right)^{1/2}\qquad \|\mathbf{A}\|_{\text{quenched}}=\frac{1}{n}\sum_{i}\left(\sum_{j}|a_{ij}|^{2}\right)^{1/2}. \tag{7}$$

Then

$$\|\mathbf{A}\|_{\text{ergodic}}\geqslant\|\mathbf{A}\|_{\text{quenched}}\geqslant|\det(\mathbf{A})|^{1/n}. \tag{8}$$

The left-hand equality applies if and only if all $x_i=(\Sigma_j|a_{ij}|^2)^{1/2}$ are equal and the right-hand equality applies if and only if all $x_i$ are equal and $\mathbf{A}^{+}\mathbf{A}$ is diagonal.

*Proof.* Let

$$\|\mathbf{A}\|_{e}^{2}=\frac{1}{n}\sum_{i}x_{i}^{2}\qquad \|\mathbf{A}\|_{q}^{2}=\left(\frac{1}{n}\sum_{i}x_{i}\right)^{2}.$$

Then the left-hand inequality follows from the convexity of $x^2$, with equality iff all $x_i$ are equal. By the geometric-mean/arithmetic-mean inequality (Beckenbach and Bellman 1983) then

$$\frac{1}{n}\sum_{i}x_{i}\geqslant\left(\prod_{i}x_{i}\right)^{1/n}.$$

We thus have

$$\|\mathbf{A}\|_{q}\geqslant\left[\prod_{i}\left(\sum_{j}|a_{ij}|^{2}\right)^{1/2}\right]^{1/n}\geqslant|\det(\mathbf{A})|^{1/n}$$

where the last step is the Hadamard inequality (Marcus and Minc 1964). The left-hand equality applies iff all $x_i$ are equal and the Hadamard inequality has equality iff either $\mathbf{A}^{+}\mathbf{A}$ is diagonal or **A** has a zero column. In the latter case $x_i=0$ for some $i$, so to have both equalities requires that $x_i=0$ for all $i$ and $\mathbf{A}=0$.

† See appendix 2.

The reason for the names 'ergodic' and 'quenched' derives from the spin glass context (Larsen 1977, 1986b). It may be supposed, at least for the purpose of a qualitative discussion, that the long-range interactions impress on each spin site, say $i$, a fluctuating magnetic field, due to the influences from all the other spins, $j \neq i$, combined. If the field averages to zero, either because of spin fluctuations or due to the random signs in $\mathbf{J}$, or both, one can presume that the magnitude of such a field energy can be obtained from the local sum of squares:

$$x_i^2 = \sum_{j(\neq i)} J_{ij}^2.$$

In this picture $x_i$ is the magnitude of the interaction energy on a specific site $i$. Thus, when all spins remain at fixed positions, these energies would, on average over the *definite* and unique configuration of positions which gives rise to the matrix $\mathbf{J}$, amount to

$$\|\mathbf{J}\|_{\text{quenched}} = \frac{1}{N_s} \sum_i x_i.$$

On the other hand, if one were to suppose that the spin $i$ moved about among the other spins in an ergodic manner, then the fluctuations would derive from the local $x_i^2$ in a variety of locations. As it is assumed that the quenched configurations are random, the ergodic interaction energy average could be obtained from the variance sum $(1/N_s)\,\Sigma_i\, x_i^2$ and would become

$$\|\mathbf{J}\|_{\text{ergodic}} = \left(\frac{1}{N_s} \sum_i x_i^2\right)^{1/2}.$$

Whether or not such explanations, or approximate solutions leading to such estimates, would be borne out by the exact solution of the statistical mechanics of the spin glass models, theorem 1 shows that these two estimates of the energy scale $g(c)$ coincide if and only if all locations have the same $x_i$. As we have emphasised, this is not the case in a quenched random configuration with $c<1$. However, the simplified models referred to above may represent situations where the sites *are* equivalent. In the model of Edwards and Anderson (1975) there are spins on every site in a lattice and the probability distribution of each $J_{ij}$ is the same, with $i$ and $j$ being lattice neighbours. In the model of Sherrington and Kirkpatrick (1975) the spins are also on the sites of a regular lattice, while the interaction is long range, but has the same probability distribution for each pair $J_{ij}$. These models may therefore correspond to a situation in which the left-hand equality in (8) applies.

In order to evaluate the ergodic norm (5) one can take advantage of the position pair correlation function: $p(r) = 3r^2/(R^3 - a_0^3)$, where, for $a_0 \leq r \leq R$, $p(r)\,\mathrm{d}r$ is the probability that a pair $(i,j)$ has a distance in the interval $[r|r+\mathrm{d}r]$ and where $R$ is the radius of a spherical volume containing $N_a$ sites each occupying a volume $4\pi a_0^3/3$, among which at random $N_s = cN_a$ sites carry a spin. With (2) then

$$\|\mathbf{J}\|_{\text{ergodic}} \rightarrow \left(3cJ_0^2 \int_1^{\infty} \mathrm{d}t \frac{\cos^2(t)}{t^4}\right)^{1/2} = k\sqrt{c}J_0 \tag{9}$$

for $N_a = (R/a_0)^3 \rightarrow \infty$, where $k = [\pi + \frac{1}{2}(1 - \cos 2 - \sin 2) - 2\,\mathrm{Si}(2)]^{1/2} = 0.429\ 18\ldots$.

In the $c$ scaling (4) the $c$ dependence of the energy scale $g(c)$ is much closer to the linear one of (3) than to the $\sqrt{c}$ of this estimate. Unfortunately, there is no analogous way to evaluate the quenched norm $\|\mathbf{J}\|_{\text{quenched}}$, which appears to be a more likely candidate, by the inequality (8), and particularly as $c \ll \sqrt{c}$ for $c \ll 1$. It would be of interest if one could evaluate the determinant of $\mathbf{J}$ at the lower bound. At present, we are not aware of any way to do this for a given $\mathbf{J}$ of the kind we are considering here. An approximate procedure has been proposed (Larsen 1977, 1986a, b), by means of which one gets

$$\|\mathbf{J}\|_{\text{quenched}} \simeq c(-\ln c)^{1/2} J_0 \tag{10}$$

which is very close to the observed $g(c)$, and it also quite accurately reacts to the damping mentioned above in the same way as is observed. This enhances the interest one may associate with the quenched norm and it would be desirable to acquire further rigorous results for it.

To exhibit the nature of the problem this poses, we shall obtain another inequality which pertains to ensembles of random matrices.

*Theorem 2.* Let $\mathbf{J} = \{J_{ij}\}$ be a member of an ensemble of real symmetric $n \times n$ matices, in which the ensemble mean of each matrix element vanishes: $\langle J_{ij}\rangle = 0$ for all $i$ and $j$, and where $\langle J_{ij}^2\rangle = \delta^2$ for all elements. The elements in a given $\mathbf{J}$ need not be statistically independent. If, for $p = 1, 2, \ldots, \langle(1/n)\mathrm{Tr}(\mathbf{J}^p)\rangle \to \int \mathrm{d}x\, w(x)x^p$ with probability 1 for $n \to \infty$, and represent the central order-$p$ moments of a continuous spectral density $w(x) = w(-x)$, where $w(x)\,\mathrm{d}x$ is the probability that an eigenvalue of $\mathbf{J}$ in the ensemble is in the interval $[x|x+\mathrm{d}x]$, and if $w(x) = 0$ for $|x| \geqslant \Delta$, then the ensemble mean of the ergodic and quenched norms satisfy the inequalities

$$\sqrt{n}\,\delta \geqslant \langle\|\mathbf{J}\|_{\text{ergodic}}\rangle \geqslant \langle\|\mathbf{J}\|_{\text{quenched}}\rangle \geqslant 1/2ew_{\mathrm{m}} \tag{11}$$

where $2e = 5.436\ 56\ldots$ and $w_{\mathrm{m}} = \max_{|x|\leqslant\Delta}(w(x), 1/\Delta)$.

*Proof.* By the ensemble mean value $\langle\,\rangle$ we mean

$$\langle\|\mathbf{J}\|\rangle = \frac{1}{\mathcal{N}}\sum_{K}^{\mathcal{N}} \|\mathbf{J}_K\|$$

for sufficiently large matrix sets $\{\mathbf{J}_K | K = 1, 2, \ldots, \mathcal{N}\}$, in the usual way. By the concavity of $\sqrt{x}$

$$\langle\|\mathbf{J}\|_e\rangle = \left\langle\left(\frac{1}{n}\sum_{i,j} J_{ij}^2\right)^{1/2}\right\rangle \leqslant \left(\frac{1}{n}\sum_{i,j}\langle J_{ij}^2\rangle\right)^{1/2} = (n\delta^2)^{1/2}$$

which proves the left-hand inequality. The middle one follows from theorem 1. For the right-hand one we need $\langle|\det(\mathbf{J})|^n\rangle$. Let $|\mathbf{J}|$ be $(\tilde{\mathbf{J}}\mathbf{J})^{1/2}$. Then by the identity $|\det(\mathbf{J})| = \exp(\mathrm{Tr}(\ln|\mathbf{J}|))$ and the convexity of $\exp(x)$ we have

$$\langle|\det(\mathbf{J})|^{1/n}\rangle = \langle\exp[(1/n)\mathrm{Tr}(\ln|\mathbf{J}|)]\rangle \geqslant \exp\langle(1/n)\mathrm{Tr}(\ln|\mathbf{J}|)\rangle.$$

Since, by the standard assumptions, with probability 1 the eigenvalues $\{\lambda_\alpha(\mathbf{J}) | \alpha = 1, 2, \ldots, n\}$ belong to a finite interval $[-\Delta|\Delta]$, one can scale $|\lambda_\alpha(\mathbf{J})|$ with $\Delta$ in order to have

$$\Delta\exp\langle(1/n)\sum_\alpha \ln(|\lambda_\alpha(\mathbf{J})|/\Delta)\rangle.$$

The possibility of zero eigenvalues of some $\mathbf{J}_K$ in the ensemble is automatically taken care of by the determinant/trace identity. The $\ln(|\lambda_\alpha(\mathbf{J})|/\Delta)$ has a convergent power series expansion in the range where $w(x)$ may have support. By the standard definition

(Wigner 1955, 1958, Arnold 1967, Edwards and Jones 1976) of $w(x)$ in terms of the order-$p$ moments, given by $\langle(1/n)\mathrm{Tr}(\mathbf{J}^p)\rangle$, we then have

$$\Delta \exp\left(\int_{-\Delta}^{\Delta} \mathrm{d}x\, w(x) \ln(|x|/\Delta)\right). \tag{12}$$

Finally we can establish a lower bound for the integral by means of the inequality of Steffensen (Beckenbach and Bellman 1983). One has, with $\tilde{w}(x) = \Delta w(\Delta x) = \tilde{w}(-x)$ and $\tilde{w}_{\mathrm{m}} = \max_{|x|\leq 1}(\tilde{w}(x), 1) = \Delta w_{\mathrm{m}}$, that

$$\exp\left(-2\int_0^1 \mathrm{d}x\, \tilde{w}(x)(-\ln(x))\right) \geq \exp\left(-2\tilde{w}_{\mathrm{m}}\int_0^{x'} \mathrm{d}x(-\ln(x))\right)$$

where

$$x' = \frac{1}{\tilde{w}_{\mathrm{m}}}\int_0^1 \mathrm{d}x\, \tilde{w}(x) = 1/2\tilde{w}_{\mathrm{m}}.$$

The integral at the lower bound equals $1/2e\tilde{w}_{\mathrm{m}}$. By the other Steffensen inequality, the integral (12) does not exceed $2/ew_{\mathrm{m}}$.

Let us first consider what this result implies for those ensembles in which the matrix elements $J_{ij}$ are independent random variables (Wigner 1955, 1958, Arnold 1967, Edwards and Jones 1976, Mehta 1967, Brody *et al* 1981). In *this* case one has the Wigner semicircle

$$w(x) = \frac{2}{\pi\Delta}[1-(x/\Delta)^2]^{1/2} \qquad \text{in} \qquad |x| \leq \Delta. \tag{13}$$

For large $n$ the range $\Delta$ increases relatively slowly:

$$\Delta = 2\sqrt{n}\,\delta. \tag{14}$$

The exact lower bound is

$$\Delta \exp\left(\frac{4}{\pi}\int_0^1 \mathrm{d}x(1-x^2)^{1/2} \ln x\right) = \Delta/2\sqrt{e}. \tag{15}$$

Our lower bound from (11) is $\Delta/2e$ and the upper bound for this integral is $2\Delta/e$. We thus find that (Wigner density)

$$\sqrt{n}\;\delta \geq \langle\|\mathbf{J}\|_{\mathrm{ergodic}}\rangle \geq \langle\|\mathbf{J}\|_{\mathrm{quenched}}\rangle \geq \sqrt{n}\,\delta/\sqrt{e}. \tag{16}$$

In the ensemble where $\mathbf{J}$ is defined by (2) for quenched random positions one may look at the distribution of a definite $J_{ij}$. The spin pair indicated will turn out to be at distances, in the *different* $\mathbf{J}$, which are distributed according to the pair correlation function $p(r)$. It is emphasised that this is entirely a result of the ensemble formation. The same calculation which gave (9) above then gives

$$\delta = kJ_0/\sqrt{N_{\mathrm{a}}} \tag{17}$$

for $N_{\mathrm{a}} \gg 1$. If one were to assume that the Wigner density (13) would apply to this ensemble, despite the fact that the elements of $\mathbf{J}$ are *not* independent, then (as $J_{ii} = 0$ is of no consequence when $n = N_{\mathrm{s}} \gg 1$) one should find that

$$k\sqrt{c}J_0 \geq \langle\|\mathbf{J}\|_{\mathrm{ergodic}}\rangle \geq \langle\|\mathbf{J}\|_{\mathrm{quenched}}\rangle \geq k\sqrt{c}J_0/\sqrt{e} \tag{18}$$

with both ensemble averages of the norms being essentially $\sqrt{c}J_0$. As regards the ergodic norm, the ensemble averaging makes no difference, as expected. On the other hand, the ensemble average appears to have a profound influence on the quenched norm *if* the assumption of the Wigner density holds.

In the derivation of the Wigner semicircular eigenvalue density it is assumed that the elements of **J** are independent random variables or it is a consequence of other assumptions inherent in the definition of the ensemble (Wigner 1955, 1958, Arnold 1967, Edwards and Jones 1976, Mehta 1967, Brody *et al* 1981). Of course, this *in itself* does not rule out the Wigner density in ensembles where the independence is not perfect, as in the present case. But hardly anything seems to be known about such ensembles.

Apart from the evidence referred to above, there exist two numerical studies, to be discussed below, which indicate that the quenched norm has a $c$ dependence essentially as in (10). At $c \ll 1$ it falls orders of magnitude below the questionable lower bound in (18). One would therefore conclude that some modification of the Wigner density has to take place. The second moment of $w(x)$ is known to be $\langle(1/n)\mathrm{Tr}(\mathbf{J}^2)\rangle = \langle\|\mathbf{J}\|^2_{\mathrm{ergodic}}\rangle = (k\sqrt{c}J_0)^2$ from (9). Most likely, therefore, a comparatively sharp peak exists in $w(x)$ at $x=0$ (which is not unheard of (Mehta 1967, Brody *et al* 1981)) to invoke $w_{\mathrm{m}}$ in the lower bound in (11) and pushes it down to a dependence which accommodates the nearly linear $c$ scaling of $\langle\|\mathbf{J}\|_{\mathrm{quenched}}\rangle$. To compensate, $w(x)$ would then need wings exceeding the spectral range of the Wigner density. These indications would seem to make the present matrix ensemble an interesting object for further study.

Freudenhammer (1977) reported the $c$ dependence of the largest eigenvalue, averaged over 10 samples of **J** with $N_{\mathrm{s}}=216$, and similarly for the eigenvalue $\lambda_{114}(\mathbf{J})$. At low $c$ the largest eigenvalue is $\sim c^{0.3}$, but the middle one nearly linear, as in (10). Both findings are consistent with the peak-and-wing picture sketched above.

In a careful numerical study Lauszus (1983) evaluated the ensemble mean of the quenched norm: $\langle\|\mathbf{J}\|_{\mathrm{quenched}}\rangle$, for systems with $N_{\mathrm{s}}$ as large as computationally feasible (up to $N_{\mathrm{s}}=500$ at $c=10^{-3}$, and to compensate for boundary effects, which is imperative with the infinite range of (2), these objects for the $i$ sum were embedded in a cubic shell holding 26 times $N_{\mathrm{s}}$ spins whose influences over the boundary was taken into account). The results are presented in Larsen (1986b, figure 2). Although the distribution of $\|\mathbf{J}\|_{\mathrm{quenched}}$ over the ensemble is strongly skewed and broadened at finite $N_{\mathrm{s}}$, due to the strong distance dependence in (2) and close pairs $(i, j)$, a statistic ($\sim$400 samples at each $c$) was obtained sufficient to demonstrate with confidence that the $c$ dependence of the ensemble mean is incompatible with $\sqrt{c}$. It could even be shown that a linear $c$ dependence is also ruled out, with confidence, in favour of a weak scaling correction consistent with the logarithmic one in (10). In this conclusion the numerical evidence agrees with a large body of experimental data obtained over the years (Larsen 1978).

Let $x_i = (\Sigma_{j(\neq i)} J_{ij}^2)^{1/2}$ be the set of $N_{\mathrm{s}}$ local quantities contained in a given matrix **J**. These $\{x_i\}$ vary randomly over the ensemble, but in a correlated fashion, which represents the interdependence of the matrix elements $\{J_{ij}\}$. Thus again

$$\|\mathbf{J}\|_{\mathrm{quenched}} = \frac{1}{N_{\mathrm{s}}}\sum_i x_i.$$

A question of importance in the theory of random matrix ensembles is whether the individual member matrices are 'ergodic', in the sense that almost every member in the limit $N_s \to \infty$ displays the same features as the ensemble mean (Mehta 1967, Brody *et al* 1981). For the simpler ensembles this is known to be the case for the eigenvalue density $w(x)$, level spacing distributions, etc. In the present ensemble it is by no means a trivial question, due to the correlations between the local $\{x_i\}$. It is not known if this 'matrix ergodicity' or 'matrix self-averaging' (which, of course, must not be confused with the spin-position ergodicity we have discussed above) holds for the eigenvalue density $w(x)$. In fact, it would seem preferable to have a proof that $w(x)$ exists at all.

However, with respect to $\|\mathbf{J}\|_{\text{quenched}}$ the numerical investigation (Lauszus 1983) gave results which can be taken as evidence for the 'matrix ergodicity' or 'self-averaging' of this quantity. The confidence limits for the distribution of $\|\mathbf{J}\|_{\text{quenched}}$ decrease with $N_s$, to a good approximation as $N_s^{-3/4}$. It may therefore be assumed that, with probability 1, as $N_s \to \infty$,

$$\frac{1}{N_s}\sum_i x_i \overset{p}{=} \langle\|\mathbf{J}\|_{\text{quenched}}\rangle = \frac{1}{N_s}\sum_i \langle x_i\rangle = \langle x_i\rangle \tag{19}$$

where it has been used that the marginal distributions of each $x_i$ on the matrix ensemble, obtained from the joint distribution of the correlated $\{x_i\}$, are identical by construction and have ensemble mean equal to $\langle x_i\rangle$. Thus for sufficiently large $N_s$ with probability 1

$$\|\mathbf{J}\|_{\text{quenched}} \overset{p}{=} \langle\|\mathbf{J}\|_{\text{quenched}}\rangle = \langle x_i\rangle. \tag{20}$$

A spin with a close neighbour will have an $x_i$ in excess of $J_0\cos(1)$, whereas one with a nearest neighbour at the *mean* nearest-neighbour distance, $r/a_0 \simeq c^{-1/3}$, will have $x_i \sim cJ_0$. In each $\mathbf{J}$ there are, on average, $N_s c$ spins in close pairs, so that in the sum $(1/N_s)\Sigma_i x_i$ they contribute about as much as all the rest. They would also account for the strongest correlations between the $\{x_i\}$. When $N_s c > 1$ the close pairs should make the spectral range of each $\mathbf{J}$ exceed $2\max_{i,j}|J_{ij}| = 2J_0$. This in itself is at variance with the conclusion (Ullah 1983), which holds for the ensembles with independent matrix elements, that there are essentially no eigenvalues outside the Wigner semicircle. If the present ensemble had a Wigner $w(x)$ its range would be given by (14) and such large eigenvalues would have to be rare. From the Geršgorin theorem (Marcus and Minc 1964) the upper bound on the largest eigenvalue is $\max_i \Sigma_j |J_{ij}|$, which would be of the same order of magnitude with respect to $c$. One cannot assert, without a detailed investigation, whether these far eigenvalues carry any weight in the *single* eigenvalues probability density $w(x)$, as $N_s \to \infty$. Let us assume that they imply $\Delta \sim J_0$. Then the upper bound in (11), together with (17), tells us that $w_m \geqslant (2ek\sqrt{c}J_0)^{-1}$. Consequently, if we take it for granted that a continuous $w(x)$ exists, we must have $w_m = \max w(x)$, unlike the Wigner case where $w_m = 1/\Delta$. Under these conditions theorem 2 thus implies that

$$\max w(x) \geqslant \frac{1}{2e\langle\|\mathbf{J}\|_{\text{quenched}}\rangle} = \frac{1}{2e\langle x_i\rangle}. \tag{21}$$

From what was inferred above about the spectrum of a typical $\mathbf{J}$, it is possible to foresee that a peak of this size, and with a corresponding width of the order $\langle\|\mathbf{J}\|_{\text{quenched}}\rangle$, exists in $w(x)$ at $x = 0$.

To evaluate the $c$ dependence of $\|\mathbf{J}\|_{\text{ergodic}}$ it was possible to use the position pair correlation function. There is no similarly obvious way to evaluate $\langle x_i\rangle$ in order to estimate $\|\mathbf{J}\|_{\text{quenched}}$. If one were to use the pair correlation function to handle the $j$ sum, he would in fact be calculating the ergodic norm, not the quenched one (Larsen

1977, 1986b). The double sum $(1/N_s)\,\Sigma_{i,j}\ldots$ in the ergodic norm simulates position ergodicity because of the original randomness in the independent positions. Nevertheless, each spin sits in a definite and unique environment, represented by its $x_i$. These $\{x_i\}$ are all fixed, by quenching, and different, by position randomness. Therefore the $j$ sum pertaining to a given site $i$ is not self-averaging (Anderson 1978), due to the selection of a definite site $i$ to start with. Thus no average is involved in a given $x_i$. In this respect quenched randomness defeats conventional statistical reasoning. At present there are no standard procedures available to replace it.

In the following we present a simple way in which the estimate (10) can be obtained from a qualitative argument. It should be emphasised that the procedure is based on physical reasoning. It was originally introduced because of its remarkable ability to account for experimental $c$ dependences, etc. The possibility that it may be a good approximation to the quenched norm is indicated by its agreement with numerical computation (Lauszus 1983). It operates with a concept, that of the 'typical neighbour density', which has no precedent. With respect to the quenched norm the position is that the only estimate we have of its $c$ dependence is (10), which is motivated by physical considerations. If the latter eventually turns out to represent the correct $c$ dependence of the quenched norm, then this norm probably is the energy scale $g(c)$. If not, (10) may merely be a good approximation to the norm and the norm a good approximation to $g(c)$.

The indication taken from (20) is that probabilistic concepts may still be of relevance, as long as they pertain to the imaginary matrix ensemble. The 'matrix ergodicity' which is then invoked has no physical counterpart (Anderson 1978). As usual, what is implied is the expectation that, with probability 1, any given *actual* **J** with sufficiently large $N_s$ will display features that may be inferred from the **J** ensemble (Mehta 1967, Brody *et al* 1981).

Let the $N_a$ atomic sites be ordered in a sequence, indicated by $k=1,2,\ldots,N_a$†, of never-decreasing distance from the centre of the volume they are embedded in. Eventually $N_a\to\infty$, with $c$ constant. For tractability we consider an interaction $J(R_{ij})$ of the form: $J_k=\pm J_0/k$, with site $i$ as centre and site $j$ at distance $r/a_0=k^{1/3}$. Let the 'typical neighbour density' be a set of weights $\{p_k^{(q)}\}$, such that $\Sigma_{k=1}^{N_a}p_k^{(q)}=N_s-1$ at $N_a\gg 1$, and for which

$$\langle x_i\rangle=\left(\sum_{k=1}^{N_a}p_k^{(q)}J_k^2\right)^{1/2}. \tag{22}$$

Defined in this way the typical neighbour density is a property of the matrix ensemble, but one which is to be regarded as 'typical' of the sites $i$ in almost all member matrices **J**.

An analogous construction to handle the double sum $(1/N_s)\,\Sigma_{i,j}J(R_{ij})^2$ would be $p_k^{(e)}=c$ for all $k$, the discrete version of the pair correlation function, which lets the probability that a site $k$ counts as a spin equal to $c$ for all sites.

We estimate that in the quenched environment a site counts as a spin with probability $c$ too, but only on the condition that it is not the first spin site encountered on the way from the central site $k=1$. The first spin should be $i$ and be excluded from the $j$ sum. Thus

$$p_k^{(q)}:=c[1-(1-c)^{k-1}]. \tag{23}$$

This typical neighbour density has a lacuna around the central site, corresponding to the fact that the majority of spins $i$ have no spins $j$ close by. The factor [ ] is the

† The index $k$ is not to be confused with the constant $k$ in (9), of course.

probability that not all sites up to and including site number $k-1$ have no spin. The lacuna excludes exactly one spin: $c\,\Sigma_{k=1}^{\infty}(1-c)^{k-1}=1$, representing the central spin $i$, and is simply the waiting-time distribution for the first spin to appear. The idea is that, in the quenched environment, the lacuna around a given spin $i$ never gets filled by spins flowing in from infinity, nor by any itineracy of spin $i$, as in the average environment. The present typical neighbour density is a discrete version of the continuous density which results from the argument first proposed by Riess and Ron (1973).

By means of standard relations for the Euler dilogarithm $L_2(c)=\Sigma_{k=1}^{\infty}c^k/k^2$ one then finds

$$\langle x_i\rangle := J_0\left(\frac{c}{1-c}[\ln c\ln(1-c)-\zeta(2)c+L_2(c)]\right)^{1/2}. \tag{24}$$

For $c<1$ we get

$$\langle x_i\rangle \sim cJ_0(-K-\ln c)^{1/2} \tag{25}$$

where $K=\zeta(2)-1=0.644\ 93\ldots$ (reasonably close to $\gamma=0.577\ 21\ldots$ which appears in the continuum version (Larsen 1977, 1986b, Riess and Ron 1973), which has an integral over $r$ instead of the sum over $k$).

For such a $\pm$ model (21) and (25) produce the guess

$$\max w(x) \geqslant \frac{1}{2ecJ_0(-K-\ln c)^{1/2}} \tag{26}$$

for $c\to 0$. Such a sharp peak in the eigenvalue density would imply that the present matrix ensemble is a remarkable object, worthy of a closer study.

From the physical viewpoint the fundamental issue in the present context is that of quenched randomness. As first emphasised by Anderson (1978), quenched randomness presents problems of a kind which cannot be solved by conventional statistical reasoning. Although, of course, ergodicity is axiomatic in conventional physical statistics, the condition of quenching eliminates this axiom. Therefore self-averaging becomes a major attraction in mathematical objects pertaining to unique quenched specimens and corresponds to what is termed 'matrix ergodicity' in the theory of random matrices. But it cannot be taken for granted *a priori.*

Whereas the unique specimen investigated may have originated historically, so to speak, out of random processes, which did, at the time, admit ergodicity, quenching 'freezes' at least some of the degrees of freedom involved in this motion. The situation thereafter becomes characterised by static 'disorder'. Quenching may be likened to the cinematographical process of 'stop action'. In fact, it almost always relies on the extremely strong temperature dependence of the timescales for the freezing degrees of freedom so that rapid cooling, more or less instantaneously, stops the flow of time in the areas concerned.

It is thus better to emphasise that the resulting disorder, or quenched randomness, presents problems inherently much more difficult than those arising in 'ordered' structures, insofar as what makes the latter tractable is a high degree of symmetry. At the present stage it is no exaggeration to claim that every rigorous result that can be obtained for quenched random situations is inherently interesting.

From the mathematical viewpoint we are concerned with random-matrix ensembles in which the matrix elements in each ensemble member may not be statistically independent random variables. As far as we are aware, little, if anything, is known about such ensembles. Not all spin glass models have this complication. The models most frequently investigated (Edwards–Anderson or Sherrington–Kirkpatrick models) are designed to avoid it. On the other hand, in the more realistic RKKY interaction models the matrix elements are dependent, because the disorder is due to spin-*position* quenched randomness, with the interaction depending in a *regular* way on the space distance between spin pairs.

We have defined so-called quenched $p$-norms (cf appendix 2) of which at least the quenched 2-norm appears to be relevant both physically (as the energy scale) and mathematically (for instance, in the random matrix context). Possibly it is their first appearance.

Our first inequality (theorem 1) is general and sets upper and lower bounds on the quenched 2-norm in terms of the more conventional ergodic 2-norm (essentially the Euclidean or Hilbert–Schmidt norm) and the determinant.

The second inequality (theorem 2) relates various ensemble means of the norms and the eigenvalue density for general random matrix ensembles. It is not assumed that the matrix elements are independent. In particular, the ensemble mean of the quenched 2-norm provides a lower bound on the maximum value of the eigenvalue density. According to our estimates, this implies a sharp central peak in the eigenvalue density for the RKKY spin glass random matrix ensemble. Provided this feature is confirmed it implies that this ensemble possesses unusual mathematical properties, which presumably may be due to the correlation between the matrix elements.

Our estimates concerning the $c$ dependence of some of the quantities, such as the quenched norm, are not claimed to be exact. They are guided by physical considerations and numerical computations and are the best that are available at present. In particular (cf appendix 1), we have clarified the phenomenon of broken space dilatation invariance for these estimates.

## Appendix 1. Remarks on space dilatation

Consider a space dilatation, in which the volume and the configuration of spin positions is kept fixed, while $a_0 \to \lambda a_0$. Then $J_{ij} = \pm J_0(R_{ij}/a_0)^{-3} \to \lambda^3 J_{ij}$. Originally the idea was to explain the $c$ scaling expressed in (3) by observing that, after this dilatation, the situation would correspond to a fractional concentration $\lambda^3 c$. If it is assumed that this implies dilatation invariance, then $\lambda^3 g(c) = g(\lambda^3 c)$. From this one would infer $g(c) \propto c$, as suggested by the original form (3) in which $c$ scaling was observed.

However, as we have pointed out (Larsen 1977, 1986a, b), dilatation invariance is not exact. The reason need not be that there is some physical length scale, not included in the present model (in practice there *are* such length scales (Larsen 1977, 1986b) but it is interesting to consider the present invariant Hamiltonian). Rather, it is the space distribution of spin positions which does not remain random after a dilatation. Suppose $\lambda < 1$. Then the original configuration, which has no pairs closer than $a_0$, cannot be random under the new conditions because it lacks pairs as close as $\lambda a_0$, which would occur in a random configuration (Larsen 1986b). Therefore, whereas it is true that the energy scale must become $\lambda^3 g(c)$, the number $\lambda^3 c$ is not the fractional concentration of a *random* configuration under the new circumstances. It is the concentration of a position distribution with anticlustering. The dilatation takes the original model out of the ensemble of randomly constructed $\mathbf{J}$ and therefore the energy scale $\lambda^3 g(c)$ is not what would be denoted by $g$ at the new concentration. The question is, therefore, to what extent the true energy scale $g$ is a functional which is sensitive to this close pair effect. If it is not, then $g(c) \propto c$ would remain a good approximation. On the other hand, it is clear that such a functional as $\|\mathbf{J}\|_{\text{ergodic}}$ is very sensitive to close pairs, which is the reason it is $\propto\sqrt{c}$. An illustration was given in Larsen (1977, 1986b). The quenched norm appears to be only marginally sensitive, to judge from the logarithmic form of our estimate (10).

These norms have peculiar properties for point-like spins. Let $\rho = 3N_s/4\pi R^3 = 3c/4\pi a_0^3$ be the space density, which is to be kept constant as $a_0 \to 0$ and $c \to 0$ in conjunction. Also, to let $J_{ij}$ be a function of absolute distance, let $J_0 = (a/a_0)^3 J_0'$, where both $a$ and $J_0'$ are to remain constant. Since, for any norm, $\|\lambda\mathbf{J}\| = |\lambda|\|\mathbf{J}\|$, this supplies a factor $(a/a_0)^3 J_0'$. The norms whose leading $c \to 0$ term is $c$ will be proportional to $\rho$ in the point limit. The ensemble means of both the ergodic and (by estimate) the quenched norms, with stronger $c \to 0$ terms, diverge. This exotic feature is due to the divergent form of $J(R)$ at $R \to 0$, which renders $\mathbf{J}$ unbounded if it has an infinitesimally close pair. The number of $a_0$-close pairs, $N_s c$, becomes zero as $c \to 0$ for fixed $N_s$. Nevertheless, the distribution over the ensemble of these norms has no finite mean, due to the members which happen to have $0^+$-close pairs. The same effect is apparent in a given $\mathbf{J}$ in the limit $N_s \to \infty$. Of course, the use of the pair correlation in $\|\mathbf{J}\|_{\text{ergodic}}$ presupposes one or the other of these conditions. The remaining option for the point limit, the norm whose ensemble mean is $\propto c$, agrees with the dilatation invariant $g(c) \propto c$, since, for point spins, the random position distribution has now been divorced from the running length scale $a_0$. But we do not know what it may be, expressed in terms of the elements $J_{ij}$ (however, cf appendix 2).

In a similar way it must be concluded that $w(x)$ is not strictly dilatation invariant, due to the missing close pairs that should maintain its spectral range when $\lambda < 1$. However, if we assume that at least the small eigenvalues are not sensitive to close pairs, then at $|x| \simeq 0$ we expect the number of eigenvalues in $|x| \leqslant \mathrm{d}x$ with $w(x) = w_c(x)$ to equal the number in $|x| \leqslant \lambda^3\,\mathrm{d}x$ with the dilated $w_{\lambda^3 c}(x)$. Thus, at $x \simeq 0$, $\lambda^3 w_{\lambda^3 c}(x) = w_c(x)$. This implies $w_c(0) \propto 1/c$. Interestingly, it is precisely what is needed to accommodate max $w(x)$ within a marginally modified lower bound, such as (26), and is therefore consistent with our estimate of $\langle\|\mathbf{J}\|_{\text{quenched}}\rangle$.

## Appendix 2. Quenched norms

It suggests itself to define the class of quenched $p$-norms ($p \geqslant 1$)

$$\|\mathbf{A}\|_p = \frac{1}{n}\sum_i \left(\sum_j |a_{ij}|^p\right)^{1/p}$$

which are norms by

$$\|\mathbf{A}+\mathbf{B}\|_p \leqslant \frac{1}{n}\sum_i \left(\sum_j (|a_{ij}| + |b_{ij}|)^p\right)^{1/p}$$

$$\leqslant \frac{1}{n}\sum_i \left[\left(\sum_j |a_{ij}|^p\right)^{1/p} + \left(\sum_j |b_{ij}|^p\right)^{1/p}\right] = \|\mathbf{A}\|_p + \|\mathbf{B}\|_p$$

by the Minkowski inequality (Beckenbach and Bellman 1983). If $\mathbf{A}$ is not Hermitian there would be a dual set analogously defined. Thus $\|\mathbf{A}\|_{\text{quenched}} = \|\mathbf{A}\|_2$. By the concavity of $x^{1/p}$ for $p \geqslant 1$ each quenched $p$-norm is less than the corresponding ergodic $p$-norm: $((1/n)\sum_{i,j}|a_{ij}|^p)^{1/p}$ (equality iff all $x_i = (\sum_j |a_{ij}|^p)^{1/p}$ are equal).

For the spin glass matrix ensemble we have no numerical evaluation, aside from the case $p=2$. With appropriate reservations, it is instructive to estimate the quenched $p$-norms by means of the typical neighbour density $\{p_k^{(q)}\}$. Thus, in the $\pm$ model,

$$\langle\|\mathbf{J}\|_p\rangle^p := cJ_0^p \sum_{k=1}^{\infty} [1-(1-c)^{k-1}]/k^p.$$

This gives rise to the Lerch transcendent or Joncquière function, of which the Euler dilogarithm is the special case for $p=2$. As there is no standard relation comparable to the one used† for $p=2$, let us proceed with the continuum version. To have exactly one spin excluded by the lacuna, one must design from $p_k^{(q)}$ the density

$$p^{(q)}(k) = \begin{cases} c\{1-\exp[-\zeta(k-1)]\} & \text{for } k \geq 1 \\ 0 & \text{for } 0 \leq k < 1 \end{cases}$$

with $\zeta = c/(1-c)$, which is easily verified by direct integration of the lacuna subtraction. Of course, this $\zeta$ agrees with $-\ln(1-c)$ at $c \to 0$. Thus we estimate

$$\langle\|\mathbf{J}\|_p\rangle^p = \left\langle\left(\sum_{j(\neq i)} |J_{ij}|^p\right)^{1/p}\right\rangle^p := cJ_0^p \int_0^\infty \mathrm{d}k\{1-\exp[-\zeta(k-1)]\}/k^p.$$

A partial integration gives

$$\langle\|\mathbf{J}\|_p\rangle^p := \frac{c}{p-1}\zeta^{p-1}e^{\zeta}\Gamma(2-p,\zeta)J_0^p \qquad \text{for } p>1.$$

For integer $p=2,3,\ldots$, there are logarithmic factors, such as was shown for $p=2$. When $p$ is not an integer we find from the standard expansion of the incomplete gamma function, for $c \to 0$, the leading terms

$$\langle\|\mathbf{J}\|_p\rangle := \begin{cases} cJ_0\left(\dfrac{\Gamma(2-p)}{p-1}\right)^{1/p} & \text{for } 1<p<2 \\ c^{2/p}J_0\left(\dfrac{1}{(p-1)(p-2)}\right)^{1/p} & \text{for } p>2, p\neq 3,4,\ldots \end{cases}$$

whereas, for instance, for $p=2$ we find $\langle\|\mathbf{J}\|_2\rangle := cJ_0[-\gamma-\ln(c)]^{1/2}$.

It is interesting to note that this estimate, for $p<2$, agrees with the estimate of $g(c) \propto c$ based on the assumption of dilatation invariance. It must be presumed that, while dilatation invariance is not exact, the quenched norms with $1<p<2$ are not sensitive to close pairs. On the other hand, at $p>2$ they apparently are sensitive, as one would expect, and the ergodic norms, which are $\propto c^{1/p}$, are even more sensitive. The physically interesting case of $p=2$ is therefore apparently marginal and has only a weak logarithmic correction to the dilatation invariance linear $c$ scaling. It is emphasised that these particular results are preliminary estimates of $c$ dependences for which a rigorous analysis would be desirable.

† For integer $p$ the relation which transforms the argument $1-c \to c$ can be expressed in terms of the Nielsen generalised polylogarithms (Kölbig 1986). Unfortunately, these useful functions are not included in the standard reference works, apart from the specialisation to the dilogarithm (the case $p=2$). In this way one can obtain the $c \to 0$ expansion, including the logarithmic correction occurring in one of the non-leading terms (the leading one when $p=2$). We shall not display these results, as they are quite analogous to what will be obtained from the continuum version of the estimate, and because we would still be short of the means to treat the cases when $p$ is not an integer.

## Appendix 3. Remarks on the range of the RKKY interaction

Some time ago we pointed out (Larsen 1977, 1978) that, in the metallic spin glass alloys, there is a clear correlation between the electrical resistivity and the RKKY interaction energy scale: the larger the resistivity, the smaller is $g(c)$. This correlation is seen in the freezing temperature $T_0$, as well as in other quantities that reflect $g(c)$. The simplest way to explain it is to recognise that electrons meeting resistance will sustain the RKKY interaction (2) only over a finite range—a phenomenon referred to as damping. The evidence in favour of this explanation is substantial (Larsen 1986b and references therein), despite occasional dissent.

Some recent work (de Châtel 1981, Zuyzin and Spivak 1986, Bulaevskii and Panyukov 1986, Jagannathan *et al* 1987) calls into question whether the RKKY interaction can become of finite range under certain circumstances. As we do ourselves (Larsen 1986b, cf footnote 22), these authors also see a possible difficulty with a model based on elastic scattering of conduction electrons. The RKKY interaction may not become damped unless one averages over the scattering centre positions. Such averaging does not take place in reality. The scattering centres are at quenched random positions, just like the spins.

However, these investigators fail to recognise that this particular model by no means represents a *necessary* condition for a finite range. Indeed, for some time there has existed an alternative model (Kaneyoshi 1975) which does not rely on such an artificial averaging device and which, nevertheless, does show the damping effect. This model simply assumes that, in the presence of disorder, plane waves are not stationary states, but have a finite lifetime. The spectral density is taken to be the Breit–Wigner Lorentzian, which can be obtained in a number of elementary ways having nothing to do with position averaging. The reality is surely more complicated, but this is adequate as a one-parameter beginning. Its RKKY interaction can be calculated exactly (Kanayoshi 1975, Larsen 1985). This model has been tested critically and extensively against the experimental evidence (Larsen 1977, 1978, 1986b). No disagreement of any substance has been found as yet and the model accounts for a considerable amount of experimental detail. An interesting recent observation (Vier and Schultz 1985) can even be said to have been predicted by it (Larsen 1986b).

Damping of the RKKY interaction is just as universal as resistivity in the conduction electrons that sustain it. So obviously it is simplest to presume that both are caused by the same kind of dissipation. That is to say, both phenomena are due to irreversibility. But averaging over scattering centre positions is, rather, a technical device. It should only be used to simulate a dissipative environment, in the fashion of the well known random phase construction. Actually the dissipation takes place by excitation of degrees of freedom with an essentially continuous spectrum. One may presume that disorder (partly) causes the existence of such degrees of freedom and that electronic collisions are responsible for their excitation.

In ordered lattices the same thing happens when electrons excite phonons. Some such mechanism is always necessary in order to have resistivity, seen as an irreversible phenomenon, whereas if such excitations are ruled out (by an energy gap and/or low temperature) the resistive dissipation does not take place. Even though elastic scattering as usual accounts for the electronic band structure in crystals.

In disordered structures the continuous spectrum probably has more strength at low energies—corresponding to excitations which may not be phonon-like. For this reason resistive dissipation may not be frozen out at low temperatures, which explains why the low-temperature resistance is much larger in disordered systems than in crystalline ones. There seems to be every reason to presume that the observed damping of the RKKY interaction is due to the same cause.

It so happens that the technical averaging over scattering centre positions may simulate this effect. But the question is really whether or not the electron self-energy has a non-zero imaginary part. To have it, one needs irreversible dissipation, and then there will be both resistivity and damping. In this respect the dissipative model is analogous to the well known optical model in nuclear physics (Bohr and Mottelson 1969). Otherwise one has a reversible quantum mechanics, even if the wavefunctions correspond to a disorderly potential with no resistivity and a long-range RKKY interaction too, according to recent reports. However interesting it is on its own account, the disorderly potential model suffices no more than models of electron propagation in periodic potentials to explain the irreversibilities that are observed in real systems.

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